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TRANSIENT RESONANCE OSCILLATIONS OF A SLOW-VARIANT SYSTEM WITH SMALL NON-LINEAR DAMPING—MODELLING AND PREDICTION

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The transient response of a single-degree-of-freedom oscillator with a slow-variant natural frequency and a small non-linear damping is under consideration. The damping is modelled as a sum of elementary power functions with respect to the system velocity. The passage through a resonance which is induced by a sweep of the excitation frequency during run-up or run-down is studied using the Krylov–Bogoljubov asymptotic method. Numerical calculations are presented to demonstrate the validity of the first asymptotic approximation. Asymptotic approximations for the maximum transient response and the corresponding excitation frequency are derived analytically in the particular case of a system with linear viscous damping. The obtained formulae are tested numerically and compared to known approximations.

1. INTRODUCTION

The widely used procedure in numerical modelling of mechanical systems capable of vibration is the modal decomposition analysis [1] which yields a decoupling of the equations of motion. The modal decomposition is possible only in the case of linear viscous damping which can be used as a crude approximation of essential dissipative mechanisms involved in a large number of systems. In fact, three qualitative different damping mechanisms should be taken into account simultaneously. There are: material damping due to spatial energy dissipation in a solid, aerodynamic damping arising from the surrounding medium and interface damping by the adjacent attachments and joints. At least some of these sources of damping, e.g., the interface damping have a non-linear character [2–4].

The usual approach in numerical models consists in the sophisticated implementation of the equivalent coefficients of linear damping using, e.g., the method of harmonic balance [3, 4] or the empirical definition of a logarithmic decrement ratio [5, 6]. In contrast to this it is suggested [7] to add non-linear terms of all types of damping directly into the modal equations after the modal decomposition of the undamped linear system. The admissibility to use this principle for a non-linear system can be explained for two main reasons. Firstly, it is assumed that the vibrating system can be treated as a weakly non-linear system, i.e.,

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damping forces are essentially less than inertia and elastic forces. In this case, the response only slightly differs from the simple harmonic during each single vibration cycle. This requirement holds in a large number of vibration problems. In this paper, this is shown with reference to turbomachine blading. Secondly, there are no closely spaced natural frequencies, i.e., the system has no internal resonances. Under these conditions, the damping and the action of external periodic forces may lead to a rapid elimination of higher harmonics of a response and to the establishment of the basic tone of vibrations. The modal decomposition for such system response is defined in reference [8] taking into account a small non-linear damping.

Rotating structures like bladed disks in air engines are driven by oscillating forces with slow-variant frequencies of their harmonics during transient operation. Varying rotational frequencies can induce essential stiffening in natural frequencies of the system due to reasons of changing gyroscopic moments and centrifugal forces [5]. In the non-resonant regime the stiffening and the system damping contribute slightly to the response. In the case of transition through a resonance the situation differs and even a small perturbation in system parameters leads to a significant change in the response. The need in accurate prediction of the influence of the system stiffening and non-linear effects on the transient resonance response is therefore an important problem. However, in most of the work done on this subject, natural frequencies are assumed to be time independent because of mathematical simplification [9–11]. Irretier and Leul [12] added, in the consideration the sweep of the natural frequency and estimated resonance amplitudes using an empirical approach based on the analysis of numerical data.

The starting point of the present work refers to the Krylov–Bogoljubov asymptotic method which is suitable for investigations of resonance regimes in non-linear slow-variant vibrating systems. In the paper, this method is applied to the investigation of a simple oscillator whose excitation frequency sweeps through the natural frequency of the system during a transient operation. The stiffening of natural frequency and damping non-linearities are taken into account using the modal equation of vibration which is proposed in reference [7].

2. DEFINITION OF THE PROBLEM

The modal equation of forced vibration of a mechanical system is considered in the following form [7] (q and T are the modal co-ordinate and time):

$$m\frac{\mathrm{d}^2q}{\mathrm{d}T^2}+\sum_{i=1}^{N}\gamma_i\left|\frac{\mathrm{d}q}{\mathrm{d}T}\right|^{n_i}\operatorname{sgn}\frac{\mathrm{d}q}{\mathrm{d}T}+m\omega^2(T)=p\cos\theta(T), \tag{1}$$

where

$$\omega(T)=\alpha T+\omega_0, \qquad \frac{\mathrm{d}\theta(T)}{\mathrm{d}T}\equiv\Omega(T)=\beta T+\Omega_0, \tag{2}$$

m is the modal mass; γ_i and n_i are the constant damping coefficients and exponents and N is the number of different damping mechanisms contributing to the total damping; $\omega(T)$ and $\Omega(T)$ are the natural frequency and the frequency of the excitation force; p is the constant amplitude of the excitation force. The time-variant natural frequency $\omega(T)$ accounts the possible stiffening effect which is typical in rotating machinery [5].

Note that at $n_i = 0$ and 1 the corresponding single dissipative terms in equation (1) describe the Coulomb dry friction and the linear viscous damping respectively. It is assumed that all other damping mechanisms may be presented by the dissipative terms with different values of exponents n_i. However, exact values of the damping exponents n_i can only be found from experimental data.

The main assumptions concerning the vibration process can be noted as follows. The natural frequency $\omega(T)$ varies slowly with time in comparison with the excitation frequency $\Omega(T)$:

$$|\alpha|/|\beta| < 1. \tag{3}$$

There are two non-dimensional small parameters δ_1 and δ_2 which characterize the smallness of the sweep rate of the excitation and damping forces respectively. Let the parameter

$$\delta_1 = \beta/\omega_r^2, \quad \delta_1 \ll 1 \tag{4}$$

describe the small non-dimensional phase acceleration where ω_r is the resonance frequency which corresponds to the crosspoint of the lines $\omega(T)$ and $\Omega(T)$:

$$\omega_r = (\beta\omega_0 - \alpha\Omega_0)/(\beta - \alpha). \tag{5}$$

The smallness of the damping forces in comparison with the inertia and elastic forces is assumed. This can be represented using the following small parameter:

$$\delta_2 \equiv \left(\frac{\gamma_k}{p} I(n_k)\right)^{1/n_k} \frac{p}{m\omega_r} = \max_i \left(\frac{\gamma_k}{p} I(n_i)\right)^{1/n_i} \frac{p}{m\omega_r}, \quad \delta_2 \ll 1, \tag{6}$$

where the normalizing factors $I(n_i)$ are defined by Gamma functions as

$$I(n_i) = \frac{2}{\sqrt{\pi}} \frac{\Gamma(\frac{1}{2}n_i + 1)}{\Gamma(\frac{1}{2}n_i + \frac{3}{2})}. \tag{7}$$

Such choice of the normalizing factors $I(n_i)$ provides the most simplest non-dimensional description of the problem due to the equality in the first approximation of energies dissipated by the origin non-linear system and the equivalent linear system per unit oscillation cycle.

The resonance oscillations caused by the transition of the frequency of excitation force $\Omega(T)$ through the value of the system natural frequency $\omega(T)$ will be considered. The following initial conditions are fixed:

$$q(0) = q_0, \quad \mathrm{d}q/\mathrm{d}T(0) = 0. \tag{8}$$

Taking into account inequality (3) and the conditions $\delta_1 \ll 1$, $\delta_2 \ll 1$, equation (1) will be reduced to a weakly non-linear equation with slowly varying parameters. Then the Krylov–Bogoljubov asymptotic method will be applied for the investigation of its approximate solution.

3. FIRST ASYMPTOTIC APPROXIMATION

To consider the non-dimensional description of the problem, the following variables

$$x = \delta(q/\hat{q}), \quad \hat{q} = p/m\omega_r^2, \quad t = \omega_r T, \tag{9}$$

where

$$\delta = \max\{\delta_1, \delta_2\} \tag{10}$$

can be introduced. In terms of these variables, equation (1) can be conveniently rewritten in the form

$$\frac{\mathrm{d}^2 x}{\mathrm{d}t^2} + \rho^2(\tau)x = \delta\left(-\sum_{i=1}^{N} \frac{r_{\gamma i}}{I(n_i)} \left|\frac{\mathrm{d}x}{\mathrm{d}t}\right|^{n_i} \operatorname{sgn}\frac{\mathrm{d}x}{\mathrm{d}t} + \cos\theta(t)\right), \tag{11}$$

where $\tau = \delta t$ is the "slowing time" and

$$\rho(\tau) = t_1\tau + 1 - \frac{r_1}{r_2}(1 - \eta_0), \quad \eta_0 = \frac{\Omega_0}{\omega_r}, \quad \frac{\mathrm{d}\theta(t)}{\mathrm{d}t} = \eta(\tau) = r_2\tau + \eta_0. \tag{12}$$

According to the definition of non-dimensional variables x and t the coefficients r_1, r_2, $r_{\gamma i}$ depend on the value δ which corresponds to the maximum small parameter involved. This maximum is unknown in advance because of the damping coefficients and the operating parameters may vary in a large range dependently on the problem considered. Therefore, two possible cases should be taken into account. In the case $\delta_1 > \delta_2$ from the relations (4), (6) and (9) follows

$$r_1 = \frac{\alpha}{\beta}, \quad r_2 = 1, \quad r_{\gamma i} = \frac{\gamma_i}{p} I(n_i)\left(\frac{p\omega_r}{m\beta}\right)^{n_i}, \tag{13}$$

and for $\delta_1 < \delta_2$ one obtains

$$r_1 = \frac{\alpha}{\beta} r_2, \quad r_2 = \frac{\beta m}{\omega_r p}\left(\frac{p}{\gamma_k I(n_k)}\right)^{1/n_k}, \quad r_{\gamma i} = \frac{\gamma_i}{p} I(n_i)\left(\frac{p}{\gamma_k I(n_k)}\right)^{n_i/n_k}. \tag{14}$$

Using relations (3), (4) and (6) the following inequalities

$$0 \leqslant r_1 < r_2 \leqslant 1, \quad 0 < r_{\gamma i} \leqslant 1, \quad i = 1, \dots, N \tag{15}$$

can be established if relations (13) or (14) are adopted.

Equation (11) corresponds to a single degree-of-freedom system with small non-linearity and slowly varying natural and excitation frequencies. An approximate solution of equation (11) can be found using the Krylov–Bogoljubov asymptotic method [8] which is suitable for the study of the resonance zone and also transitions through the latter from the non-resonance zone. According to this method, the first approximation of the solution of equation (11) can be represented in the form

$$x = a\cos(\theta + \varphi), \tag{16}$$

where the functions a and φ are postulated as solutions of a system of differential equations according to the following statement.

If the functions a and φ are the solutions of the system of ordinary differential equations

$$\begin{aligned} \frac{\mathrm{d}a}{\mathrm{d}t} &= -\frac{\delta a}{2}\left(\frac{r_1}{\rho(\tau)} + \sum_{i=1}^{N} r_{\gamma i} a^{n_i - 1} \rho^{n_i - 1}(\tau)\right) - \frac{\delta \sin\varphi}{\rho(\tau) + \eta(\tau)}, \\ \frac{\mathrm{d}\varphi}{\mathrm{d}t} &= \rho(\tau) - \eta(\tau) - \frac{\delta\cos\varphi}{a(\rho(\tau) + \eta(\tau))}, \end{aligned} \tag{17}$$

where the parameter δ is sufficiently small, then an approximate solution of equation (11) can be represented by expression (16) with an accuracy of δ^2 for a finite interval $t \in [0, L]$, $L > (1 - \eta_0)/\delta$.

The main advantages of the Krylov–Bogoljubov first approximation are:

- The equations of the first approximation provide the definition of an amplitude envelope and a phase of the response. The maximum of the transient resonance response can be found from the smooth envelope function $a(t)$ instead of an analysis of the fast oscillating function $x(t)$. These equations can be reduced to qudrature formulae [8] in the case of linear viscous damping $n_i = 1$.
- If monofrequency vibrations are considered, it is possible to show that the resulting response may be approximated as a linear superposition of quasi-normal responses where each quasi-normal response is defined by an equation of type (11). In reference [13] this decomposition is considered in detail and also some assumptions concerning the constitutive equations of damping forces are discussed.

3.1. BACKGROUND OF ASYMPTOTIC APPROACH

An asymptotic approach, which is used for the investigation of the system response, can be used only if the non-dimensional parameters δ_1 and δ_2 are

TABLE 1

Damping coefficients for turbine blades

Type of blade	δ_e
One-piece disk blade	0·004–0·008
Fir-tree rooted blade	0·012–0·02
Cooled blade	0·02–0·03
Blade with root damper	0·036–0·048
Blade with shroud damper	0·016–0·048

sufficiently small. To verify these requirements, consider reasonable values of the parameters δ_1 and δ_2 with reference to turbomachine blading.

Using relation (4) the value δ_1 can be estimated if the angular acceleration of the nuzzle excitation β and the modal resonance frequency ω_r are known. In air-engines the first modal frequency of turbine blades varies in the range of 150–800 Hz. The run-up time from the rest to the maximum angular velocity of 200 rpm may in extreme cases be 1·8–2 s. Taking into account that the number of nozzles is 30–40, the angular acceleration and the resonance frequency can be estimated as $\beta = 4000\ 1/\mathrm{s}^2$ and $\omega_r = 940\ 1/\mathrm{s}$. Substituting these values into relation (3), the maximum non-dimensional acceleration can be found: $\delta_1 = 0{\cdot}0045$.

The maximum non-dimensional damping δ_2 can be estimated by the coefficient δ_e of the linear viscous damping $\delta_2 \leqslant \delta_e$ using, for example, the technique of an equivalent linearization of the stationary response. For the turbine rotor blades the ranges of equivalent damping coefficients δ_e from references [5, 6] are summarized in Table 1.

To verify the validity of the first asymptotic approximation (17), numerical simulations are performed for the values of non-dimensional small parameters δ_1 and δ_2 which are vary in the range estimated in this section.

4. APPROXIMATIONS

Now consider the response of a system with the linear viscous damping. In this case it is possible to evaluate an approximate asymptotic solution in quadratures. In the following analysis these quadrature formulae are investigated by means of the complex contour integration.

4.1. EXACT ASYMPTOTIC SOLUTION

The non-dimensional modal equation (11) for the system with linear viscous damping $N = 1$, $n_1 = 1$ has the form

$$\frac{\mathrm{d}^2 x}{\mathrm{d}t^2} + r_\gamma \delta \frac{\mathrm{d}x}{\mathrm{d}t} + \rho^2(\tau) x = \delta \cos\theta(t), \tag{18}$$

$$\rho(\tau) = r_1 \tau + 1 - r_1(1 - \eta_0), \quad \frac{\mathrm{d}\theta}{\mathrm{d}t} \equiv \eta(\tau) = \tau + \eta_0. \tag{19}$$

An approximate asymptotic solution of this equation can be found in the form (16) where according to equations (17) the functions a and φ are solutions of the following system of equations:

$$\frac{\mathrm{d}a}{\mathrm{d}t} = -\frac{\delta a}{2}\left(\frac{r_1}{\rho(\tau)} + r_\gamma\right) - \frac{\delta \sin\varphi}{\rho(\tau) + \eta(\tau)},$$

$$\frac{\mathrm{d}\varphi}{\mathrm{d}t} = \rho(\tau) - \eta(\tau) - \frac{\delta \cos\varphi}{a(\rho(\tau) + \eta(\tau))}. \tag{20}$$

Using the change of variable

$$z = a\mathrm{e}^{\mathrm{i}\varphi}, \quad (\mathrm{i} = \sqrt{-1}) \tag{21}$$

the last system of equations reduces to a linear differential equation of the first order with respect to the variable z which can be easily integrated. Using this integral the quadrature of the amplitude as a function of the excitation frequency is obtained:

$$a^2(\eta) = k_1 \frac{\mathrm{e}^{-r_\gamma \eta}}{r_1(\eta - 1) + 1} |x_0 + k_2 J|^2, \tag{22}$$

where

$$J = \int_{\eta_0}^{\eta} g_1(\lambda)\mathrm{e}^{(1/\delta)g_2(\lambda)}\,\mathrm{d}\lambda \tag{23}$$

and $g_1(\lambda)$, $g_2(\lambda)$, k_1, k_2 are

$$g_1(\lambda) = \frac{\sqrt{r_1(\lambda - 1) + 1}}{(1 + r_1)\lambda + 1 - r_1}\,\mathrm{e}^{1/2 r_\gamma \lambda}, \tag{24}$$

$$g_2(\lambda) = \frac{i}{2}(1 - r_1)(\lambda - 1)^2, \tag{25}$$

$$k_1 = (1 - r_1(1 - \eta_0))\mathrm{e}^{r_\gamma \eta_0}, \tag{26}$$

$$k_2 = \frac{-i}{\sqrt{1 - r_1(1 - \eta_0)}}\,\mathrm{e}^{-(\mathrm{i}/2\delta)(1 - r_1)(\eta_0 - 1)^2 - (1/2) r_\gamma \eta_0}. \tag{27}$$

Thus, the calculation of the amplitude is reduced to the evaluation of the integral J. In the next section asymptotic estimations of this integral are considered in terms of elementary functions.

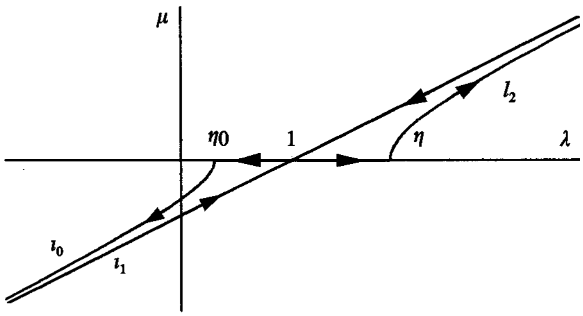

Figure 1. The integration contour on the plane $z = \lambda + i\mu$.

4.2. SADDLE-POINT METHOD

The investigation of the integral J is difficult due to fast transient oscillations of the function to be integrated. However, the complex contour integration gives a successful method of solution, if we apply the saddle-point method [14] which is suitable for an asymptotic integration of the function with a sufficiently large exponent. According to the saddle-point method the contour of integration should consists of level lines l_k along which the imaginary part of the function

$$g_2(z) = g_2(\lambda + i\mu) \tag{28}$$

is constant.

Using this approach the integration contour is defined as a combination of the segment $[\eta_0, \eta]$ and lines l_0, l_1, l_2 which are parametrically defined by the relations

$$l_k : z_k = \lambda + i\mu_k, \quad k = 0, 1, 2, \tag{29}$$

where

$$\mu_0 = -\sqrt{(\lambda - 1)^2 - (\eta_0 - 1)^2}, \quad \lambda \in (-\infty; \eta_0], \tag{30}$$

$$\mu_1 = \lambda - 1, \quad \lambda \in (-\infty;\ \infty), \tag{31}$$

$$\mu_2 = \sqrt{(\lambda - 1)^2 - (\eta - 1)^2}, \quad \lambda \in [\eta; \infty). \tag{32}$$

Performing an integration along the contour shown in Figure 1 and taking into account that there are no singularities enclosed within this contour we obtain by the Cauchy theorem the following expression:

$$J = I_1 - I_0 - I_2, \tag{33}$$

$$I_k = \int_{l_k} g_1(z) e^{(1/\delta) g_2(z)} \, dz, \quad k = 0, 1, 2. \tag{34}$$

4.3. ASYMPTOTIC EVALUATION OF THE INTEGRALS

The integrals I_0, I_1, I_2 can be easily estimated up to the terms of the order δ using the Taylor series of functions to be integrated. Consider these expansions in powers of the small parameter δ in the vicinities of corresponding bounding points.

According the relations (29), (30), and (34) the integral I_0 can be written in the form

$$I_0 = \mathrm{e}^{(\mathrm{i}/2\delta)(1-r_1)(\eta_0-1)^2} \int_{-\infty}^{\eta_0} g_1(z_0) \frac{\mathrm{d}z_0}{\mathrm{d}\lambda} \mathrm{e}^{-(1/\delta)(1-r_1)(\lambda-1)\mu_0} \mathrm{d}\lambda. \quad (35)$$

Changing of the variable in integration

$$\tau = (1 - r_1)(\lambda - 1)\mu_0 \quad (36)$$

and considering the Taylor series of the function to be integrated in the small right-hand vicinity $0 < \tau < \varepsilon$, integral (35) can be estimated as

$$I_0 = \frac{1}{\sqrt{2}(1-r_1)} \mathrm{e}^{(\mathrm{i}/2\delta)(1-r_1)(\eta_0-1)^2} \int_0^{\varepsilon} \left(\frac{\mathrm{i}\sqrt{2} g_1(\eta_0)}{1-\eta_0} + O(\tau) \right) \mathrm{e}^{-(\tau/\delta)} \mathrm{d}\tau + o(\delta). \quad (37)$$

Evaluating this integral gives the asymptotic approximation of the integral I_0:

$$I_0 = \mathrm{i}\mathrm{e}^{(\mathrm{i}/2\delta)(1-r_1)(1-\eta_0)^2} \frac{g_1(\eta_0)\delta}{(1-r_1)(1-\eta_0)} + o(\delta). \quad (38)$$

Now consider the integral

$$I_1 = (1 + \mathrm{i}) \int_{-\infty}^{\infty} g_1(z_1) \mathrm{e}^{-(\mathrm{i}/\delta)(1-r_1)(\lambda-1)^2} \mathrm{d}\lambda. \quad (39)$$

By analogy with an evaluation of the integral I_0, the following new variable τ of integration

$$\tau = (1 - r_1)(\lambda - 1)^2 \quad (40)$$

together with the corresponding Taylor series can be applied which yields

$$I_1 = \frac{(1+\mathrm{i})}{\sqrt{1-r_1}} \int_0^{\varepsilon} (g_1(1) + O(\sqrt{\tau})) \frac{\mathrm{e}^{-\tau/\delta}}{\sqrt{\tau}} \mathrm{d}\tau + o(\delta). \quad (41)$$

Evaluating this integral gives the following approximation:

$$I_1 = \frac{\sqrt{\pi}}{4} (1 + \mathrm{i}) \mathrm{e}^{(1/2)r_y} \sqrt{\frac{\delta}{1-r_1}} + o(\delta). \quad (42)$$

Finally, consider an asymptotic estimation of the integral

$$I_2 = e^{(i/2\delta)(1-r_1)(\eta-1)^2} \int_\eta^\infty g_1(z_2) \frac{dz_2}{d\lambda} e^{-(1/\delta)(1-r_1)(\lambda-1)\mu_2} d\lambda. \tag{43}$$

Taking the change of the variables u, p:

$$u = \frac{1}{\delta}(1-r_1)(\lambda-1)\mu_2, \quad p = \frac{1-r_1}{2\delta}(\eta-1)^2 \tag{44}$$

in integral (43) gives

$$I_2 = \frac{1}{2}\sqrt{\frac{\delta}{1-r_1}}\, e^{ip} \int_0^\infty g_1(z_2) \frac{(A(p,u)+iB(p,u))}{\sqrt{p^2+u^2}} e^{-u} du, \tag{45}$$

where

$$g_1(z_2) = g_1\left(1 + \sqrt{\frac{\delta}{1-r_1}}(B(p,u)+iA(p,u))\right), \tag{46}$$

$$A(p,u) = \sqrt{-p+\sqrt{p^2+u^2}}, \quad B(p,u) = \sqrt{p+\sqrt{p^2+u^2}}. \tag{47}$$

Considering the following expansion of the right-hand side of relation (46)

$$g_1(z_2) = \frac{1}{2} e^{(1/2)r_\gamma}\left(1 + \frac{1}{2}(r_\gamma - 1)\sqrt{\frac{\delta}{1-r_1}}(B(p,u)+iA(p,u))\right) + o(\sqrt{\delta}) \tag{48}$$

and substituting this series into integral (45) gives the asymptotic approximation of the integral I_2:

$$I_2 = \frac{1}{4}\sqrt{\frac{\delta}{1-r_1}}\, e^{(1/2)r_\gamma + ip}\left(M(p) + i(r_\gamma - 1)\sqrt{\frac{\delta}{1-r_1}}\right) + o(\delta), \tag{49}$$

where

$$M(p) = \int_0^\infty \frac{(A(p,u)+iB(p,u))}{\sqrt{p^2+u^2}} e^{-u} du. \tag{50}$$

In the following analysis the derived approximations (38), (42), and (49) will be used to evaluate the maximum amplitude during the transient excitation.

4.4. APPROXIMATION OF THE MAXIMUM AMPLITUDE

The maximum amplitude can be calculated by quadrature formula (22) where, according to relation (33), the integral J is evaluated by means of approximations of the integrals I_0, I_1, and I_2, which are given by the expressions (38), (42), and (49). To evaluate the integral I_2 it is necessary to know the excitation frequency $\eta = \eta_{max}$ corresponding to the maximum amplitude a_{max}. In the present work η_{max} is assumed to be the smallest root of the equation

$$\frac{\mathrm{d}a^2(\eta)}{\mathrm{d}\eta} = 0, \tag{51}$$

which is larger than 1 or, in other terms, the largest peak of an amplitude occurs in a small right-hand vicinity of the resonance point $\eta = 1$.

Differentiating relation (22), the last equation can be written as

$$k_1 \frac{g_1(\eta)}{\mathrm{d}\eta} |x_0 + k_2 J|^2 + k_1 g_1(\eta) \left((x_0 + k_2^* J^*) \frac{\mathrm{d}J}{\mathrm{d}\eta} + (x_0 + k_2 J) \frac{\mathrm{d}J^*}{\mathrm{d}\eta} \right) = 0, \tag{52}$$

where asterisks denote complex conjugated values.

To evaluate the left-hand side of this equation the following relations

$$\frac{\mathrm{d}J}{\mathrm{d}\eta} = g_3(\eta)\mathrm{e}^{\mathrm{i}p}, \quad \frac{\mathrm{d}J^*}{\mathrm{d}\eta} = g_3(\eta)\mathrm{e}^{-\mathrm{i}p} \tag{53}$$

can be found differentiating integral (23) and taking into account the second relation (44).

According to relation (33), where the integrals I_0, I_1, and I_2 are given by the expressions (38), (42), and (49), the integral J can be represented in the form

$$J = \frac{1}{2}\sqrt{\frac{\delta}{1 - r_1}}\, \mathrm{e}^{(1/2)r_\gamma} \left(\mathrm{e}^{\mathrm{i}(\pi/4)}\sqrt{2\pi} + \frac{1}{2}\mathrm{e}^{\mathrm{i}p} M(p) \right) + O(\delta). \tag{54}$$

Taking into account the relations (53) and (54) and omitting the terms $O(\delta)$, equation (52) can be approximated by the following equation with respect to p:

$$\cos\left(p - \frac{\pi}{4}\right) = \frac{1}{2\sqrt{2\pi}} \int_0^\infty \frac{A(p, u)}{\sqrt{p^2 + u^2}}\, \mathrm{e}^{-u}\, \mathrm{d}u. \tag{55}$$

Solving this equation numerically one obtains $p \approx 2{\cdot}32725$ which together with the second relation (44) leads to the following approximation of the critical frequency of maximum response:

$$\eta_{max} \approx 1 + 2{\cdot}157 \sqrt{\frac{\delta}{1 - r_1}}. \tag{56}$$

Substituting the obtained value p into integral (49) and collection the integrals (38), (42), and (49) in relation (22) yields the desired approximation of the maximum transient response.

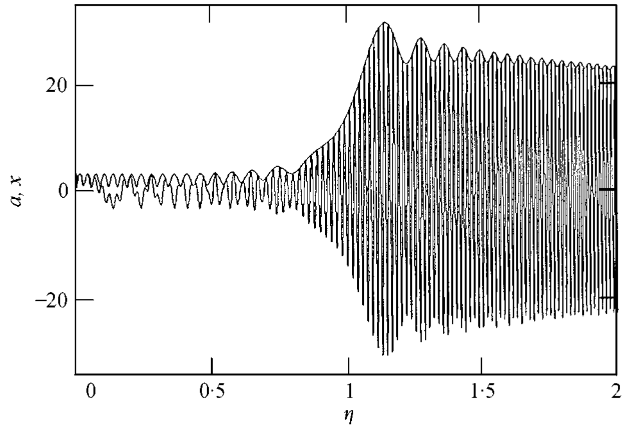

Figure 2. Response and amplitude versus excitation frequency, $\delta = 0{\cdot}003$, $r_1 = 0{\cdot}333$, $r_2 = 1$, $r_\gamma = 1{\cdot}667$, $n = 2{\cdot}0$.

The analysis of the obtained approximate formulae can be summarized as:

- It is proved theoretically that the resonance frequency η_{max} does not depend on the damping in the first approximation. In the particular case of a system with constant natural frequency the derived approximations are in correspondence with the results presented by Fearn an Millsaps [10] and Markert and Seidler [11].
- Oscillations of the maximum transient response with respect to a change in the sweep rate of the natural frequency are established. These oscillations arise due to the influence of the exponent of the function I_0 given by relation (38). This property can be explained as a stiffening induced instability.

5. NUMERICAL RESULTS

A series of calculations was performed to verify the validity of the first approximation (16) for some reasonable values of the parameters δ, r_1, $r_{\gamma i}$. Typical results of a numerical integration of exact equation (11) (response x) and its first approximation (16) (amplitude a) are shown in Figures 2 and 3 where only the single non-linear damping term ($N = 1$) was taken into account. The amplitude a and the response x are represented as functions of excitation frequency η. A good accuracy of amplitude envelopes can be observed especially in the resonance region.

Figure 4 shows the maximum transient response a_{max} which is calculated at different values of the damping exponent n (parameters δ, r_1 are fixed). Each of the three curves corresponds to calculations with the same value r_γ. Boxes represent the values a_{max} which are calculated from the exact equation (11), crosses correspond to the first approximation (16). Figure 5 represents the values of excitation frequency η_{max} corresponding to the maximum transient response a_{max}. These calculations are

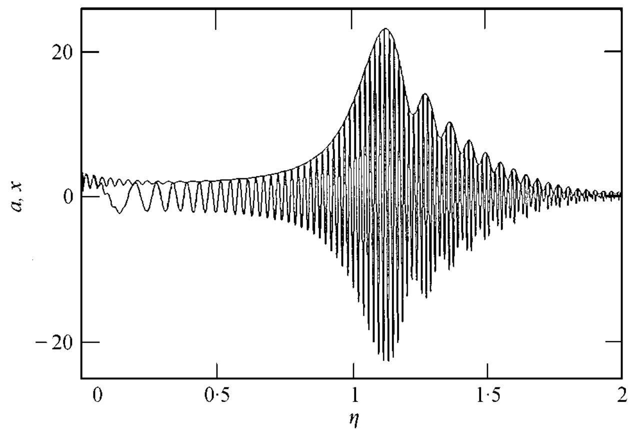

Figure 3. Response and amplitude versus excitation frequency, $\delta = 0{\cdot}003$, $r_1 = 0{\cdot}333$, $r_2 = 1$, $r_\gamma = 1{\cdot}667$, $n = 0{\cdot}6$.

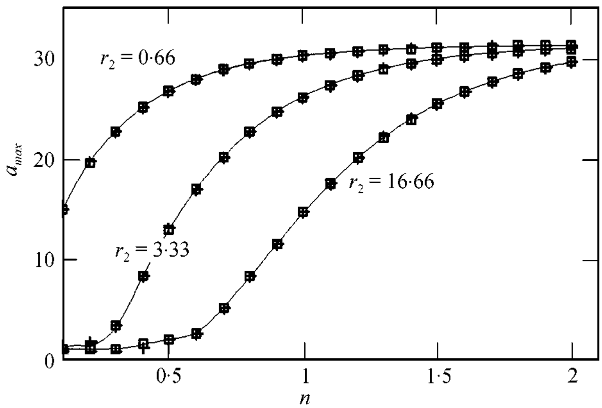

Figure 4. a_{max} versus n, $\delta = 0{\cdot}003$, $r_1 = 0{\cdot}333$, $r_2 = 1$.

performed at the same values of parameters as for the series in Figure 3. Negligible small errors of a_{max} and the corresponding frequencies were found for all values of the parameters of calculation in the intervals considered. This indicates the validity of the first approximation for the maximum transient response investigations.

In order to verify the derived formula of a maximum response, some numerical results are given in Figures 6–8 for reasonable values of the parameters δ, r_1, r_γ. Typical results of the comparative analysis of the derived formula for the maximum transient amplitude a_{max}, the exact solution and the approximation suggested by Irretier and Leul [12] are shown in Figure 6. In contrast to known approximations, evident oscillations of a_{max} can be correctly predicted using the new approximation. These oscillations have an exponentially increasing character with respect to the increase of the sweep-rate parameter r_1 and can be observed in Figure 7 which presents exact values of the maximum transient response calculated numerically.

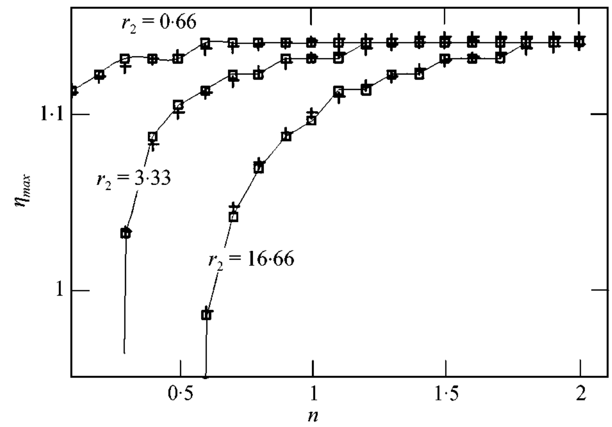

Figure 5. η_{max} versus n, $\delta = 0{\cdot}003$, $r_1 = 0{\cdot}333$, $r_2 = 1$.

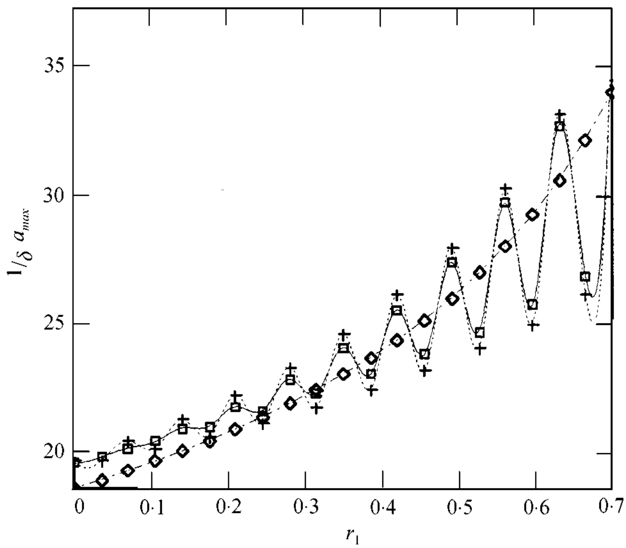

Figure 6. Comparison of maximum amplitudes, $\delta = 0{\cdot}005$, $r_\gamma = 0{\cdot}092$. —◇— Irretier and Leul 1995; —□— exact solution; —+— new approximation

Figure 8 shows an error $\varDelta$ of the obtained approximate formulae

$$\varDelta = \frac{|a_{max} - a'_{max}|}{a_{max}}, \tag{57}$$

where a_{max} is the exact maximum amplitude which was calculated by numerical integration of equation (11) and a'_{max} is the approximation which was calculated by substitution of the approximate critical frequency (56) into the quadrature formula of the amplitude (22). The clearly observed peak in Figure 8 indicates that the critical frequency of a maximum response η_{max} and the resonance frequency $\eta = 1$

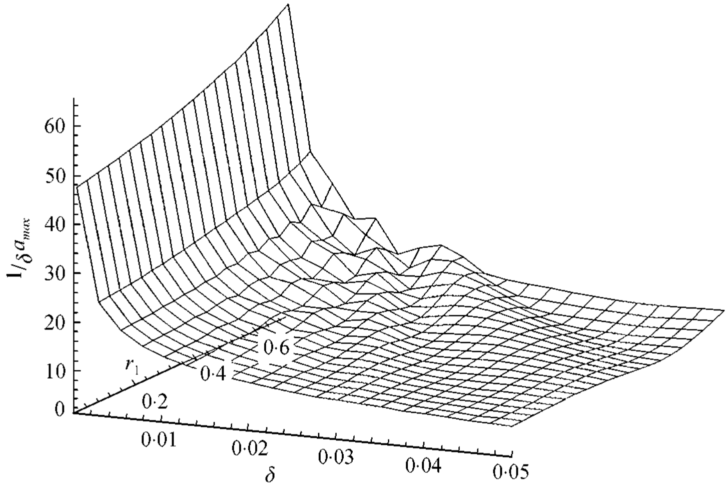

Figure 7. Exact maximum amplitudes of the response, $\delta r_\gamma = 0{\cdot}01$.

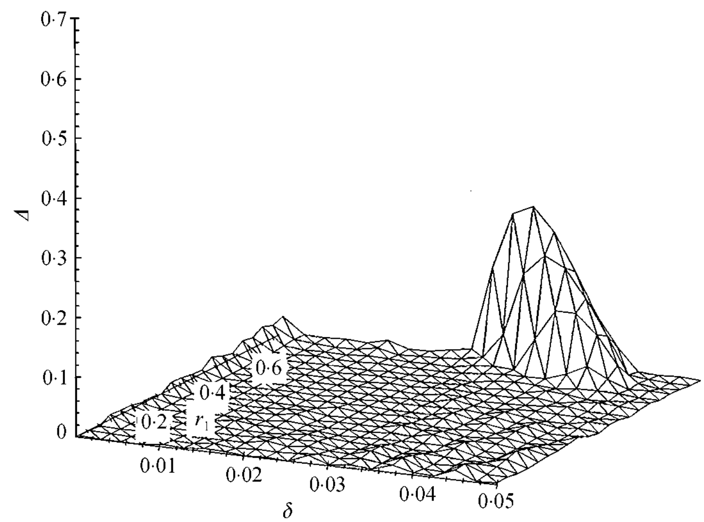

Figure 8. Error of approximate formulae, $\delta r_\gamma = 0{\cdot}01$.

are not closely spaced for all values of the system parameters from this region, therefore η_{max} can not be found using equation (51). In this case an amplitude reaches its maximum value instantly after starting due to the sufficiently large value of the integral I_0 at the beginning time interval.

6. CONCLUDING REMARKS

An applicability of the first asymptotic approximation in transient vibration problems is established taking into account the sources of non-linear damping and a possible stiffening in natural frequencies.

The non-linear damping law considers a sum of elementary power functions with respect to the modal velocity (1). Basing on physical reasons two small non-dimensional parameters are defined to characterize the smallness of damping forces and a slow run-up during the transient operation. Introducing the non-dimensional response function the problem is reduced to the investigation of a weakly non-linear equation with a slowly varying natural frequency. The harmonic balance method is applied to obtain equations of the first asymptotic approximation, which are tested numerically for reasonable values of the system parameters. Upper estimations to damping coefficients and sweep rates of an excitation are obtained with reference to turbomachine blade vibrations.

Theoretically based approximation of the maximum transient repsonse which takes into account the dependencies on sweep rates, damping and initial conditions is worked out for a system with a linear viscous damping. This provides a simple analysis of qualitative effects arising due to time dependent properties of a system and can be used to estimate correctly the modal quantities in model updating algorithms. A good agreement with numerical simulation data is established. The oscillations of the maximum amplitude with respect to a change in the stiffening parameter of the natural frequency may be a source of the circular scattering of maximum amplitudes in structures with rotational symmetry during a tansient operation.

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